arXiv:hep-th/9609240v1   30 Sep 1996

Two–Dimensional BF Model
Quantized in the Axial Gauge

S. Emery[1] ⊢, M. Krüger⊣, J. Rant⊣, M. Schweda⊣ and T. Sommer[2] ⊣

⊢ Département de Physique Théorique, Université de Genève
24 quai E. Ansermet, CH – 1211 Genève 4 (Switzerland)

⊣ Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstraße 8-10, A-1040 Wien (Austria)

Abstract. The two–dimensional topological BF model is quantized in the axial gauge. We show that this theory is trivially ultraviolet finite and that the usual infrared problem of the propagator of the scalar field in two dimensions is replaced by an easily solvable long distances problem inherent to the axial gauge. It will also be shown that contrarily to the 3–dimensional case, the action principle cannot be completely replaced by the various Ward identities expressing the symmetries of the model; some of the equation of motion are needed.

1   Introduction

The main property of Topological Field Theories (TFT) [1] is the fact that their observables are of topological nature, i.e. they only depend on the global properties of the manifold on which the theory is defined. Therefore, these kind of theories exhibit remarkable ultraviolet finiteness properties due to the lack of any physical, *i.e.* metric dependent observables. In particular, the two–dimensional BF model treated here provides an example of a fully finite quantum field theory.

The history of TFT's is closely related to the relationship between the problems that arise in the research of physical systems and the ensuing mathematical methods that are needed for their solution. A well known example is the relation between the work of Donaldson concerning the study of the topology of four dimensional manifolds [2, 3] and its description as a Topological Yang–Mills theory due to Witten [4]. A further example is the study of the knot and link invariants in the case of three–dimensional Chern–Simons theory [5].

The Topological Yang–Mills theory and the Chern–Simons theory are examples of two distinct classes of TFT's, the former belonging to the Witten type – *i.e.* the whole action is a BRST variation – and the latter being of the Schwarz type – the action splits into an invariant part and a BRST variation term.

There exists a further type of Schwarz class TFT's: the Topological BF models [6]. They constitute the natural extension of the Chern–Simons model in an arbitrary number of spacetime dimensions. These models describe the coupling of an antisymmetric tensor field to the Yang–Mills field strength [7], [8].

The aim of the present work is the quantization of the two–dimensional BF model in the axial gauge. It is motivated by similar works done for the Chern–Simons model [9] and the BF model [10] both in three spacetime dimensions. The axial gauge is particularly interesting for these two models since in this gauge these two theories are obviously ultraviolet finite due to the complete absence of radiative corrections. Since for the two–dimensional case we are interested in, the choice of the axial gauge allows us to overcome the usual infrared problem which occurs in the propagator of the dimensionless fields [11], it is interresting to see wether the ultraviolet finitness properties are also present.

On the other hand, we have already shown that for the three–dimensional Chern-Simons and BF models [9], [10], the symmetries completely define the theory, *i.e.* the quantum action principle is no more needed. The latter property relies on the existence of a topological linear vector supersymmetry [12] besides the BRST invariance [13, 14]. The generators of the latter together with the one of the BRST–symmetry form a superalgebra of the Wess–Zumino type [15, 16], which closes on the translations [17]. Then the associated Ward identities can be solved for the Green's functions, exactly and uniquely.

As this linear vector supersymmetry is also present in two–dimensions, it is natural to ask ourself wether all the Green's functions of the two–dimensional model are also uniquely determined by symmetry considerations. We will answer this question by the negative. The consistency relations let some class of Green's functions undetermined and therefore one has to use some of the equations of motion.

The present work is organized as follows. The model is introduced in section 2 and its symmetries are discussed. In section 3 we will check the consistency conditions between the various symmetries and thus recover almost all the equations of motion for the theory. Section 4 is devoted to the derivation of the Green's functions keeping in mind that some are consequences of symmetries only whereas some others are solution of the field equations. At the end we draw some conclusions.

2 The two–dimensional BF model in the axial gauge

The classical BF model in two spacetime dimensions is defined by

$$S_{\text{inv}} = \frac{1}{2} \int_{\mathcal{M}} d^2x \, \varepsilon^{\mu\nu} F^a_{\mu\nu} \phi^a \quad . \tag{2.1}$$

One has to stress that the action (2.1) does not depend on a metric $g_{\mu\nu}$ which one may introduce on the arbitrary two–dimensional manifold $\mathcal{M}$. In this paper, $\mathcal{M}$ is chosen to be the flat Euclidean spacetime[1]. $\varepsilon^{\mu\nu}$ is the totally antisymmetric Levi–Civita–tensor[2]. The field strength $F^a_{\mu\nu}$ is related to the Yang–Mills gauge field A^a_μ by the structure equation

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \quad , \tag{2.2}$$

and the B field, traditionally denoted by ϕ for the two dimensional case, is a scalar field.

The fields $(A^a_\mu, F^a_{\mu\nu}, \phi^a)$ belong to the adjoint representation of a simple compact gauge group $\mathcal{G}$. The corresponding generators T^a obey

$$[T^a, T^b] = f^{abc} T^c \quad , \quad Tr(T^a T^b) = \delta^{ab} \quad , \tag{2.3}$$

f^{abc} are the totally antisymmetric structure constants of $\mathcal{G}$. The field–strength satisfies the Bianchi–identity

$$(D_\rho F_{\mu\nu})^a + (D_\mu F_{\nu\rho})^a + (D_\nu F_{\rho\mu})^a = 0 \quad , \tag{2.4}$$

---

[1]$\mathcal{M}$ has the metric $\eta_{\mu\nu} = \text{diag}\,(+1, +1)$.

[2] The tensor $\varepsilon^{\mu\nu}$ is normalized to $\varepsilon^{12} = \varepsilon_{12} = 1$ and one has $\varepsilon^{\mu\nu}\varepsilon_{\sigma\tau} = \delta^\mu_\sigma \delta^\nu_\tau - \delta^\mu_\tau \delta^\nu_\sigma$.

where the covariant derivative D_μ in the adjoint representation is

$$(D_\mu \cdot)^a = (\partial_\mu \cdot)^a + f^{abc} A_\mu^b (\cdot)^c \quad . \tag{2.5}$$

The equations of motion are

$$\frac{\delta S_{\text{inv}}}{\delta \phi^a} = \frac{1}{2} \varepsilon^{\mu\nu} F_{\mu\nu}^a = 0 \quad , \tag{2.6}$$

$$\frac{\delta S_{\text{inv}}}{\delta A_\mu^a} = \varepsilon^{\mu\nu} (D_\nu \phi)^a = 0 \quad . \tag{2.7}$$

Eq. (2.6) implies the vanishing curvature condition,

$$F_{\mu\nu}^a = 0 \quad , \tag{2.8}$$

and from eq. (2.7) follows that the scalar field is confined on the hypersphere

$$\phi^a \phi^a = \text{const.} \quad . \tag{2.9}$$

The action (2.1) is invariant under the infinitesimal gauge transformations

$$\begin{aligned}
\delta A_\mu^a &= -(\partial_\mu \theta^a + f^{abc} A_\mu^b \theta^c) = -(D_\mu \theta)^a \quad , \\
\delta \phi^a &= f^{abc} \theta^b \phi^c \quad ,
\end{aligned} \tag{2.10}$$

where θ^a is the infinitesimal local gauge parameter.

We choose to work in the axial gauge

$$n^\mu A_\mu^a = 0 . \tag{2.11}$$

and without loss of generality, we fix the axial gauge vector to be

$$n^\mu = (n^1, n^2) = (1, 0) \quad . \tag{2.12}$$

Following the usual BRST procedure, one introduces a ghost, an antighost and a Lagrange multiplier field $(c^a, \bar{c}^a, b^a)$ in order to construct the gauge fixed action S in a manifestly BRST–invariant manner,

$$S = S_{\text{inv}} + S_{\text{gf}} \quad , \tag{2.13}$$

$$\begin{aligned}
S_{\text{gf}} &= \int d^2x \, s(\bar{c}^a n^\mu A_\mu^a) \\
&= \int d^2x \, (b^a n^\mu A_\mu^a + \bar{c}^a n^\mu \partial_\mu c^a + \bar{c}^a f^{abc} n^\mu A_\mu^b c^c) \quad . \tag{2.14}
\end{aligned}$$

The nilpotent BRST–transformations are given by

$$
\begin{aligned}
sA_\mu^a &= -(D_\mu c)^a \quad , \\
s\phi^a &= f^{abc} c^b \phi^c \quad , \\
sc^a &= \tfrac{1}{2} f^{abc} c^b c^c \quad , \\
s\bar{c}^a &= b^a \quad , \\
sb^a &= 0 \quad , \\
s^2 &= 0 \quad .
\end{aligned}
\tag{2.15}
$$

Note that the gauge fixing term S_{gf} is not metric independent and therefore not topological.

The canonical dimensions and ghost charges of the fields are collected in table 1, where the canonical dimension of the gauge direction n^μ is zero.

	A_μ^a	ϕ^a	c^a	$\bar{c}^a$	b^a
canonical dimension	1	0	0	1	1
ghost charge ($Q_{\Phi\Pi}$)	0	0	1	-1	0

Table 1:   Canonical dimension and ghost charge of the fields

Before investigating further let us make some comments concerning the axial gauge. It is well known that this choice does not fix the gauge completely. Indeed (2.13) is still invariant under gauge transformations of the same type as (2.10) but where the gauge parameter θ^a depends only on x^1. This residual gauge invariance will play an important role in the sequel.

The action (2.13) possesses, besides the BRST–symmetry (2.15) and the scale invariance, an additional linear vector supersymmetry. In order to derive the vector supersymmetry transformations, one considers the energy–momentum tensor $T_{\mu\nu}$ of the theory. Since the invariant part of the action (2.1) is metric–independent, the improved energy–momentum tensor is an exact BRST–variation,

$$
T_{\mu\nu} = s\Lambda_{\mu\nu} \quad , \tag{2.16}
$$

where $\Lambda_{\mu\nu}$ is given by

$$
\Lambda_{\mu\nu} = \eta_{\mu\nu} \left(\bar{c}^a \, n^\lambda A_\lambda^a \right) - \bar{c}^a \left(n_\mu A_\nu^a + n_\nu A_\mu^a \right) \quad . \tag{2.17}
$$

Eq. (2.16) explicitly shows the unphysical character of the topological BF model. Using the functional form for the equations of motion

$$
\frac{\delta S}{\delta A_\nu^a} = \varepsilon^{\nu\mu} D_\mu \phi^a + n^\nu (b^a - f^{abc} \bar{c}^b c^c) \quad , \tag{2.18}
$$

4

$$\frac{\delta S}{\delta \phi^a} = \frac{1}{2} \varepsilon^{\mu\nu} F_{\mu\nu}^a \quad , \tag{2.19}$$

$$\frac{\delta S}{\delta c^a} = n^\mu (D_\mu \bar{c})^a \quad , \tag{2.20}$$

$$\frac{\delta S}{\delta \bar{c}^a} = n^\mu (D_\mu c)^a \quad , \tag{2.21}$$

and the gauge condition

$$\frac{\delta S}{\delta b^a} = n^\mu A_\mu^a \quad , \tag{2.22}$$

one gets for the divergence of (2.17)

$$\partial^\nu \Lambda_{\nu\mu} = \varepsilon_{\mu\nu} n^\nu \bar{c}^a \frac{\delta S}{\delta \phi^a} - A_\mu^a \frac{\delta S}{\delta c^a} + \partial_\mu \bar{c}^a \frac{\delta S}{\delta b^a} + \text{total deriv.} \quad . \tag{2.23}$$

An integration over the two–dimensional spacetime yields the Ward–identity of the linear vector supersymmetry,

$$\mathcal{W}_\mu S = 0 \quad , \tag{2.24}$$

where

$$\mathcal{W}_\mu = \int d^2x \left(\varepsilon_{\mu\nu} n^\nu \bar{c}^a \frac{\delta}{\delta \phi^a} - A_\mu^a \frac{\delta}{\delta c^a} + \partial_\mu \bar{c}^a \frac{\delta}{\delta b^a} \right) \tag{2.25}$$

and the transformations read

$$\begin{aligned}
\delta_\mu A_\nu^a &= 0 \quad , \\
\delta_\mu \phi^a &= \varepsilon_{\mu\nu} n^\nu \bar{c}^a \quad , \\
\delta_\mu c^a &= -A_\mu^a \quad , \\
\delta_\mu \bar{c}^a &= 0 \quad , \\
\delta_\mu b^a &= \partial_\mu \bar{c}^a \quad .
\end{aligned} \tag{2.26}$$

Remark: It can be easily verified that S_{inv} and S_{gf} are not separately invariant under (2.26); only the combination (2.13) is invariant.

The generator δ_μ of the linear vector supersymmetry and the BRST–operator s form a graded algebra of the Wess–Zumino type which closes on–shell on the translations,

$$\begin{aligned}
\{s, \delta_\mu\} A_\nu^a &= \partial_\mu A_\nu^a - \varepsilon_{\mu\nu} \frac{\delta S}{\delta \phi^a} \quad , \\
\{s, \delta_\mu\} \phi^a &= \partial_\mu \phi^a + \varepsilon_{\mu\nu} \frac{\delta S}{\delta A_\nu^a} \quad , \\
\{s, \delta_\mu\} \psi^a &= \partial_\mu \psi^a \quad , \quad \forall \psi^a \in \{c^a, \bar{c}^a, b^a\} \quad .
\end{aligned} \tag{2.27}$$

Moreover, the following algebraic relations hold,

$$\{\delta_\mu, \delta_\nu\} = 0 \quad , \quad [\delta_\mu, \partial_\nu] = 0 \quad . \tag{2.28}$$

5

3 Consequences of the symmetries

We will now discuss the implications of the symmetries discussed above independently of the action (2.13). This means that we are going to consider only the gauge fixing condition, the Ward identity for the vector supersymmetry and the Slavnov–Taylor identity and look for their consistency independently from the field equations (2.18)–(2.21). Since we are ultimately interested in deriving all the Green's functions of the theory, let us first rewrite all these functional identities in term of the generating functional for the connected Green's functions. We will see later that due to the absence of loop graphs for this theory, it is sufficient to restrict ourself to the tree approximation, *i.e.* the Legendre transformation of the classical action:

$$\frac{\delta Z^{\mathrm{c}}}{\delta j^{\mu\,a}} = A^a_\mu \quad , \qquad \frac{\delta S}{\delta A^a_\mu} = -\,j^{\mu\,a} \quad , \tag{3.1}$$

$$\frac{\delta Z^{\mathrm{c}}}{\delta j^a_\phi} = \phi^a \quad , \qquad \frac{\delta S}{\delta \phi^a} = -\,j^a_\phi \quad , \tag{3.2}$$

$$\frac{\delta Z^{\mathrm{c}}}{\delta j^a_c} = c^a \quad , \qquad \frac{\delta S}{\delta c^a} = \;\; j^a_c \quad , \tag{3.3}$$

$$\frac{\delta Z^{\mathrm{c}}}{\delta j^a_{\bar c}} = \bar c^a \quad , \qquad \frac{\delta S}{\delta \bar c^a} = \;\; j^a_{\bar c} \quad , \tag{3.4}$$

$$\frac{\delta Z^{\mathrm{c}}}{\delta j^a_b} = b^a \quad , \qquad \frac{\delta S}{\delta b^a} = -\,j^a_b \quad , \tag{3.5}$$

$$Z^{\mathrm{c}}[j^{\mu\,a}, j^a_\phi, j^a_c, j^a_{\bar c}, j^a_b] = S[A^a_\mu, \phi^a, c^a, \bar c^a, b^a]+ \tag{3.6}$$
$$+ \int d^2x \left(j^{\mu\,a} A^a_\mu + j^a_\phi \phi^a + j^a_c c^a + j^a_{\bar c} \bar c^a + j^a_b b^a \right) \quad ,$$

where the canonical dimensions and ghost charges of the classical sources $j^{\mu\,a}$, j^a_ϕ, j^a_c, $j^a_{\bar c}$, j^a_b are given in table 2.

	$j^{\mu\,a}$	j^a_ϕ	j^a_c	$j^a_{\bar c}$	j^a_b
canonical dimension	1	2	2	1	1
ghost charge $(Q_{\Phi\Pi})$	0	0	-1	1	0

Table 2:   Dimensions and ghost charges of the sources

The gauge condition (2.22) now reads

$$n^\mu \frac{\delta Z^{\mathrm{c}}}{\delta j^{\mu\,a}} = -\,j^a_b \quad , \tag{3.7}$$

6

and the linear vector supersymmetry Ward identity (2.24)

$$\mathcal{W}_\mu Z^{\mathrm{c}} = \int d^2x \left[- \varepsilon_{\mu\nu} n^\nu j_\phi^a \frac{\delta Z^{\mathrm{c}}}{\delta j_{\bar{c}}^a} - j_c^a \frac{\delta Z^{\mathrm{c}}}{\delta j^{\mu a}} - j_b^a \partial_\mu \frac{\delta Z^{\mathrm{c}}}{\delta j_{\bar{c}}^a} \right] = 0 \quad . \qquad (3.8)$$

The BRST–invariance is formally expressed by the Slavnov–Taylor identity

$$\mathcal{S}(Z^{\mathrm{c}}) = \int d^2x \left(j^{\mu a}[(D_\mu c)^a] \cdot Z^{\mathrm{c}} - j_\phi^a [f^{abc} c^b \phi^c] \cdot Z^{\mathrm{c}} + \right.$$
$$\left. + j_c^a [\tfrac{1}{2} f^{abc} c^b c^c] \cdot Z^{\mathrm{c}} + j_{\bar{c}}^a \frac{\delta Z^{\mathrm{c}}}{\delta j_b^a} \right) = 0 \quad . \qquad (3.9)$$

We have used the notation $[\mathcal{O}] \cdot Z^c$ for the generating functional of the connected Green's functions with the insertions of a local field polynomial operator $\mathcal{O}$. Usually, such insertions must be renormalized and their renormalization is controlled by coupling them to external sources. But in the case of the axial gauge, these insertions are trivial due to the fact that the ghost fields decouple from the gauge field as we will see later.

In order to analyze the consequences of these functional identities, let us begin by the projection of the supersymmetry Ward–identity along the axial vector n^μ

$$n^\mu \mathcal{W}_\mu Z^{\mathrm{c}} = - \int d^2x \, j_b^a \underbrace{\left(- j_c^a + n^\mu \partial_\mu \frac{\delta Z^{\mathrm{c}}}{\delta j_{\bar{c}}^a} \right)}_{X^a} = 0 \quad , \qquad (3.10)$$

where the gauge condition (3.7) has been used. Locality, scale invariance and ghost charge conservation imply that X^a is a local polynomial in the classical sources j and their functional derivatives $\delta/\delta j$ of dimension 2 and ghost charge -1. The most general form for X^a may depends on a further term

$$X^a = - j_c^a + n^\mu \partial_\mu \frac{\delta Z^{\mathrm{c}}}{\delta j_{\bar{c}}^a} + z^{abc} j_b^b \frac{\delta Z^{\mathrm{c}}}{\delta j_{\bar{c}}^c} \quad , \qquad (3.11)$$

provided z^{abc} is antisymmetric in a and b. The latter is thus proportional to the structure constants f^{abc}. By substituting the general form (3.11) into (3.10)

$$\int d^2x \, j_b^a \, X^a = 0 \quad , \qquad (3.12)$$

one gets

$$n^\mu \partial_\mu \left(\frac{\delta Z^{\mathrm{c}}}{\delta j_{\bar{c}}^a} \right) - \alpha f^{abc} j_b^b \frac{\delta Z^{\mathrm{c}}}{\delta j_{\bar{c}}^c} = j_c^a \qquad (3.13)$$

which, up to the undetermined coefficient α, corresponds to the antighost equation (2.20). In order to complete this identification, let us use the fact that, in any gauge theory with a linear gauge fixing condition, there exist a ghost equation which in

our case follows from the Slavnov–Taylor identity differentiated with respect to the source for $n^\mu A_\mu$. Indeed this gives

$$n^\mu \partial_\mu \Big(\frac{\delta Z^{\mathrm{C}}}{\delta j_c^a} \Big) - f^{abc} j_b^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_c^c} = j_{\bar{c}}^a \quad , \tag{3.14}$$

which corresponds to (2.21). At this level, it is clear that consistency between (3.13) and (3.14) fixes the value $\alpha = 1$. Therefore, the equations of motion for the ghost sector (3.13), (3.14) are direct consequences of the symmetries. Furthermore, these equations show that the ghosts only couple to the source J_b and therefore, it is possible to factorize out the effect of the ghost fields from the Slavnov–Taylor identity (3.9). The latter is thus replaced by a local gauge Ward identity

$$-\partial_\mu j^{\mu\, a} - n^\mu \partial_\mu \Big(\frac{\delta Z^{\mathrm{C}}}{\delta j_b^a} \Big) + f^{abc} j^{\mu\, b} \frac{\delta Z^{\mathrm{C}}}{\delta j^{\mu c}} + f^{abc} j_\phi^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_\phi^c} +$$
$$+ f^{abc} j_c^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_c^c} + f^{abc} j_{\bar{c}}^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar{c}}^c} + f^{abc} j_b^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_b^c} = 0 \quad . \tag{3.15}$$

The Ward identity expressing the invariance of the theory under the residual gauge symmetry discussed in sect. 2 corresponds to the integration of (3.15) with respect to x^1. The residual Ward identity reads

$$\int_{-\infty}^{\infty} dx^1 \left\{ f^{abc} j^{\mu\, b} \frac{\delta Z^{\mathrm{C}}}{\delta j^{\mu c}} + f^{abc} j_\phi^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_\phi^c} + f^{abc} j_c^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_c^c} + \right.$$
$$\left. + f^{abc} j_{\bar{c}}^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar{c}}^c} + f^{abc} j_b^{;b} \frac{\delta Z^{\mathrm{C}}}{\delta j_b^c} - \partial_2 j^{2\, a} \right\} = 0 \quad . \tag{3.16}$$

It is important to notice that this step is plagued by the bad long distance behaviour of the field b. Indeed, passing from (3.15) to (3.16) would imply

$$\int_{-\infty}^{\infty} dx^1 \partial_1 \Big(\frac{\delta Z^{\mathrm{C}}}{\delta j_b^a} \Big) = 0 \tag{3.17}$$

which is not the case as it can be shown using the solutions given latter. This is the usual IR problem of the axial gauge. In order to enforces (3.17), one substitutes

$$\frac{\delta}{\delta J_b} \leftrightarrow e^{-\varepsilon (x^1)^2} \frac{\delta}{\delta J_b} \qquad (\varepsilon > 0) \tag{3.18}$$

which corresponds to a damping factor for the b–field along the x^1 direction, and takes the limit $\varepsilon \to 0$ at the end. It turns out that this simple substitution is sufficient in order to get (3.17) and that the limit can be done trivially. The check is straightforward for all the Green's functions and therefore is left to the reader.

For the gauge sector, let us begin with the transversal component of the supersymmetry Ward identity and the ghost equation (3.14) written as functional

operator acting on Z^{c}

$$\mathcal{W}^{\text{tr}} Z^{\text{c}} \equiv \mathcal{W}_2 Z^{\text{c}} = \int d^2x \left[-j_\phi^a \frac{\delta}{\delta j_{\bar{c}}^a} - j_c^a \frac{\delta}{\delta j^{2\,a}} - j_b^a \partial_2 \frac{\delta}{\delta j_{\bar{c}}^a} \right] Z^{\text{c}} \;\; = \;\; 0, \quad (3.19)$$

$$\mathcal{G}^a Z^{\text{c}} = \left(\partial_1 \frac{\delta}{\delta j_c^a} - f^{abc} j_b^b \frac{\delta}{\delta j_c^c} \right) Z^{\text{c}} \;\; = \;\; j_{\bar{c}}^a. \quad (3.20)$$

Then, a direct calculation shows that the consistency condition

$$\{\mathcal{W}_2, \mathcal{G}^a\} \, Z^{\text{c}} = \mathcal{W}_2 j_{\bar{c}}^a \quad\quad\quad (3.21)$$

is in fact equivalent to the equation of motion for A_2 (2.19)

$$\left(\partial_1 \frac{\delta}{\delta j^{2\,a}} - f^{abc} j_b^b \frac{\delta}{\delta j^{2\,c}} \right) Z^{\text{c}} = j_\phi^a - \partial_2 j_b^a \quad . \quad\quad (3.22)$$

Up to now, we recover the equations of motion for c, $\bar{c}$ and A_2 as consistency conditions between the various symmetries but, contrary to the higher dimensional case, we did not get any information about the dynamic of ϕ.

To clarify this point, let us consider the three–dimensional BF model [10]. In this case, it is well known that the field B is a one–form and therefore is invariant under the so–called reducible symmetry

$$s B_\mu^a = - (D_\mu \psi)^a \quad\quad\quad (3.23)$$

where ψ is a zero–form. To fix this extra symmetry one needs, besides the usual Yang-Mills ghost, antighost and Lagrange multiplier fields $(c, \bar{c}, b)$, a second set of such fields $(\psi, \bar{\psi}, d)$ related to (3.23). The consequence is that we exactly double the number of terms because of the exact similarity between the Yang-Mills part $(A_\mu, c, \bar{c}, b)$ and the reducible part $(B_\mu, \psi, \bar{\psi}, d)$. Therefore, it is easy to convince one-self that we recover the two antighost equations and, by considering the consistency conditions of the same type as (3.21), that we are in a position to construct the two equations of motion for A_μ and B_μ[3]. For the two–dimensional case investigated here the field ϕ is a 0–form and does not exhibit any reducible symmetry of the type (3.23). The invariance rather than covariance of ϕ is the basic difference between the two–dimensional BF model and the higher dimensional case.

Finally, in order to clarify the present two–dimensional situation, let us collect all the functional identities we have got from symmetry requirements alone. These are the gauge condition (3.7), the antighost equation (3.13) with $\alpha = 1$, the ghost equation (3.14), the transversal component of the supersymmetry Ward identity (3.19), the field equation for A_2 (3.22), the local Ward identity (3.15) and the residual

---

[3] For a detailed discussion, see eqs. (4.3)–(4.5) in [10]

gauge Ward identity (3.16) which are respectively given by[4]

$$\frac{\delta Z^{\mathrm{C}}}{\delta j^{1\,a}} = -j_b^a \tag{3.24}$$

$$\partial_1\Big(\frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar c}^a}\Big) - f^{abc} j_b^b \frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar c}^c} = j_c^a \tag{3.25}$$

$$\partial_1\Big(\frac{\delta Z^{\mathrm{C}}}{\delta j_c^a}\Big) - f^{abc} j_b^b \frac{\delta Z^{\mathrm{C}}}{\delta j_c^c} = j_{\bar c}^a \tag{3.26}$$

$$\int d^2 x \left\{ -j_\phi^a \frac{\delta}{\delta j_{\bar c}^a} - j_c^a \frac{\delta}{\delta j^{2\,a}} - j_b^a \partial_2 \frac{\delta}{\delta j_{\bar c}^a} \right\} Z^{\mathrm{C}} = 0 \tag{3.27}$$

$$\Big(\partial_1 \frac{\delta}{\delta j^{2\,a}} - f^{abc} j_b^b \frac{\delta}{\delta j^{2\,c}}\Big) Z^{\mathrm{C}} = j_\phi^a - \partial_2 j_b^a, \tag{3.28}$$

$$\partial_1\Big(\frac{\delta Z^{\mathrm{C}}}{\delta j_b^a}\Big) + f^{abc} j^{\mu\,b} \frac{\delta Z^{\mathrm{C}}}{\delta j^{\mu\,c}} + f^{abc} j_\phi^b \frac{\delta Z^{\mathrm{C}}}{\delta j_\phi^c} +$$

$$+ f^{abc} j_c^b \frac{\delta Z^{\mathrm{C}}}{\delta j_c^c} + f^{abc} j_{\bar c}^b \frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar c}^c} + f^{abc} j_b^b \frac{\delta Z^{\mathrm{C}}}{\delta j_b^c} = \partial_\mu j^{\mu\,a} \tag{3.29}$$

$$\int_{-\infty}^{\infty} dx^1 \left\{ f^{abc} j^{\mu\,b} \frac{\delta Z^{\mathrm{C}}}{\delta j^{\mu\,c}} + f^{abc} j_\phi^b \frac{\delta Z^{\mathrm{C}}}{\delta j_\phi^c} + f^{abc} j_c^b \frac{\delta Z^{\mathrm{C}}}{\delta j_c^c} +\right.$$

$$\left. + f^{abc} j_{\bar c}^b \frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar c}^c} + f^{abc} j_b^b \frac{\delta Z^{\mathrm{C}}}{\delta j_b^c} - \partial_\mu j^{\mu\,a} \right\} = 0 \quad . \tag{3.30}$$

As mentioned before, the procedure described above fails to produce an important relation. Indeed, the equation of motion (2.18)

$$\Big(\varepsilon^{\mu\nu}\partial_\nu \frac{\delta}{\delta j_\phi^a} + n^\mu \frac{\delta}{\delta j_b^a}\Big) Z^{\mathrm{C}} + \varepsilon^{\mu\nu} f^{abc} \frac{\delta Z^{\mathrm{C}}}{\delta j^{\nu\,b}} \frac{\delta Z^{\mathrm{C}}}{\delta j_\phi^c} - n^\mu f^{abc} \frac{\delta Z^{\mathrm{C}}}{\delta j_c^b} \frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar c}^c} = -j^{\mu\,a} \tag{3.31}$$

is not a consequence of the symmetries and therefore, has to be derived from the action (2.13). More precisely, the $\mu = 1$ component of (3.31) is the field equation for ϕ

$$\Big(\partial_1 \frac{\delta}{\delta j_\phi^a} - f^{abc} j_b^b \frac{\delta}{\delta j_\phi^c}\Big) Z^{\mathrm{C}} = -j^{2\,a} \tag{3.32}$$

and the $\mu = 2$ component of (3.31) is the field equation for b

$$\Big(\frac{\delta}{\delta j_b^a} - \partial_2 \frac{\delta}{\delta j_\phi^a}\Big) Z^{\mathrm{C}} - f^{abc} \frac{\delta Z^{\mathrm{C}}}{\delta j^{2\,b}} \frac{\delta Z^{\mathrm{C}}}{\delta j_\phi^c} - f^{abc} \frac{\delta Z^{\mathrm{C}}}{\delta j_c^b} \frac{\delta Z^{\mathrm{C}}}{\delta j_{\bar c}^c} = -j^{1\,a} \tag{3.33}$$

which is nothing else than an integrability condition.

Remarks:

All the functional identities obtained by symmetry considerations are linear in the quantum fields and, hence, they are not affected by radiative corrections. This

---

[4]We will now systematically substitute (2.12).

linearization originates from the topological supersymmetry. On the other hand, the equation for b is quadratic and may causes problems. This point will be treated later.

4 Calculation of the Green's functions, perturbative finiteness

We will now derive the solution to the set of equations (3.24) – (3.31). In turn, this will prove that the tree approximation (3.6) corresponds to the exact solution.

4.1 Solution of the Gauge Condition

We already emphasized that the axial gauge allows for the factorization of the ghost sector. This is illustrated by the solution for the gauge condition (3.24)

$$\langle A_1^a(x)\, b^b(y)\rangle = -\,\delta^{ab}\delta^{(2)}(x-y) \quad . \tag{4.1}$$

which is the only non–vanishing Green's function containing A_1^a.

4.2 Solution of the ghost sector

Let us first differentiate the antighost equation (3.25) with respect to the source j_c^a. This leads to[5]

$$\partial_1 \left. \frac{\delta^2 Z^C}{\delta j_{\bar{c}}^a(x)\, \delta j_c^b(y)} \right|_{j=0} = -\,\delta^{ab}\delta^{(2)}(x-y) \tag{4.2}$$

A subsequent integration yields the propagator

$$\langle \bar{c}^a(x)\, c^b(y)\rangle = \delta^{ab}\left[-\theta(x^1-y^1)\delta(x^2-y^2) + F(x^2-y^2)\right] \tag{4.3}$$

where θ is the step function,

$$\theta(x-y) = \begin{cases} 1 & , \quad (x-y) > 0 \\ 0 & , \quad (x-y) < 0 \end{cases} \quad . \tag{4.4}$$

---

[5]Our conventions for functional derivatives of even and/or odd objects are

$$\frac{\delta}{\delta C}\int AB = \int \left(\frac{\delta A}{\delta C}B + (-1)^{\deg(A)\deg(B)}\frac{\delta B}{\delta C}A\right)$$

and $F(x^2 - y^2)$ is a function of $x^2 - y^2$ due to translational invariance, and with canonical dimension 1. As the introduction of any dimensionfull parameters in our theory will spoil its topological character and that we work in the space of tempered distributions[6], we are left with

$$\langle \bar{c}^a(x)\, c^b(y) \rangle = -\delta^{ab} \left[\theta(x^1 - y^1) + \alpha \right] \delta(x^2 - y^2) \qquad (4.5)$$

The analogous calculation starting form the ghost equation (3.26) leads to the value

$$\alpha = -\frac{1}{2} \quad . \qquad (4.6)$$

for the integration constant as a consequence of the Fermi statistics for the ghost fields and the $c \leftrightarrow \bar{c}$ invariance of the theory. This implies the principal value prescription for the unphysical pole $(n^\mu k_\mu)^{-1}$ in the Fourier–transform of the ghost–antighost propagator.

It is important to note that contrarily to the Landau gauge [11], the ghost–antighost propagator is infrared regular in the axial gauge.

For the higher order Green's functions, the basic recurence relation is obtained by differentiating (3.25) with respect to the most general combination of the sources $(\delta^{(n+m+1)}/(\delta j_c)(\delta j_b)^n(\delta j_\varphi)^m)$ for $n + m \geq 1$ and $\varphi \in \{A_2, \phi, c, \bar{c}\}$. This gives the following recursion relation over the number of b–fields

$$\partial_1 \left\langle \bar{c}^a(x)c^b(y)b^{c_1}(z_1)\ldots b^{c_n}(z_n)\varphi^{d_1}(v_1)\ldots\varphi^{d_m}(v_m) \right\rangle = \qquad (4.7)$$

$$= \sum_{k=1}^{n} f^{ac_kc}\, \delta^{(2)}(x - z_k) \times$$

$$\times \left\langle \bar{c}^c(z_k)c^b(y)b^{c_1}(z_1)\ldots \widehat{b^{c_k}(z_k)}\ldots b^{c_n}(z_n)\varphi^{d_1}(v_1)\ldots\varphi^{d_m}(v_m) \right\rangle$$

where $\widehat{\Phi}$ denotes the omission of the field Φ in the Green's functions.

Let us first look at the case $n = 0$ where (4.7) reduces to

$$\partial_1 \left\langle \bar{c}^a(x)\, c^b(y)\, \varphi^{d_1}(v_1)\ldots\varphi^{d_m}(v_m) \right\rangle = 0 \qquad (4.8)$$

The solution is

$$\left\langle \bar{c}^a(x)\, c^b(y)\, \varphi^{d_1}(v_1)\ldots\varphi^{d_m}(v_m) \right\rangle = F(\xi_k) \qquad 1 \leq k \leq M \qquad (4.9)$$

where ξ_k, stands for the $M = 1 + 2m + \frac{m}{2}(m-1)$ differences $\{x^2 - y^2,\ x^2 - v_i^2,\ v_i^2 - v_j^2,\ y^2 - v_i^2\}$, $1 \leq i,j \leq m$ due to translational invariance. The absence of any dependance on the coordinate 1 comes from the fact that any Green's function

---

[6]Any expression of the type $1/(x^2 - y^2)$ exhibits short distance singularities and therefore needs the introduction of a dimensionfull UV substraction point in order to give it a meaning.

which does not involve b–fields obeys an homogeneous equation similar to (4.8) for all its arguments.

Under the same assumptions as for (4.5), $F(\xi_k)$ has the general form

$$F(\xi_k) \sim \delta(\xi_k) \tag{4.10}$$

where the coefficients are either constants or proportional to $\ln(\frac{\xi_k}{\xi_{k'}})$ since this is the only combination which do not break scale invariance. Using now canonical dimension arguments (c.f. Tab. 1), conservation of the ghost charge and residual gauge invariance (3.30), one gets

$$\langle (\bar{c})^{m_1} (c)^{m_2} (A_\mu)^{m_3} (\phi)^{m_4} \rangle = 0 \qquad \forall \, \{m_1, m_2, m_3, m_4\} \neq \{1, 1, 0, 0\} \tag{4.11}$$

The next step concerns the Green's functions which involves b–fields. As a consequence of (4.11), the unique starting point for the recurence (4.7) is the two point function $\langle c\,\bar{c} \rangle$ (4.5). Thus (4.7) solves to the recurence relation

$$\left\langle \bar{c}^a(x)c^b(y)b^{c_1}(z_1)\ldots b^{c_n}(z_n) \right\rangle = \tag{4.12}$$

$$= \sum_{k=1}^{n} f^{ac_k c}[\theta(x^1 - z_k^1) + \alpha^{(n)}]\delta(x^2 - z_k^2) \left\langle \bar{c}^c(z_k)c^b(y)b^{c_1}(z_1)\ldots \widehat{b^{c_k}(z_k)}\ldots b^{c_n}(z_n) \right\rangle$$

for $n \geq 1$. The integration constants $\alpha^{(n)}$ are also fixed by the Fermi statistics of the ghost fields to be

$$\alpha^{(n)} = -\frac{1}{2} \quad , \quad \forall n \tag{4.13}$$

and these solutions correspond to tree graphs

$$\langle \bar{c}^a(x)\, c^b(y)\, b^{c_1}(z_1)\,\ldots\, b^{c_n}(z_n) \rangle = \tag{4.14}$$

$$= -\sum_{k=1}^{n} f^{ec_k c} \left\langle \bar{c}^a(x)\, c^e(z_k) \right\rangle \left\langle \bar{c}^c(z_k)\, c^b(y)b^{c_1}(z_1)\,\ldots\, \widehat{b^{c_k}(z_k)}\,\ldots b^{c_n}(z_n) \right\rangle$$

since (3.25,3.26) are linear in the quantum fields. Thus, this justify the tree approximation (3.6) for this sector.

4.3   Solution of the gauge sector

Although we already know that the symmetries fail to produce an important relation for this sector, let us look how far we can go in the determination of the Green's functions when taking into account only the symmetries for the model. The most fruitful approach is based on the transversal component of the supersymmetry Ward identity (3.27).

The two points functions are found by differentiating (3.27) with respect to $\delta^{(2)}/\delta j^2 \delta j_c$, $\delta^{(2)}/\delta j_\phi \delta j_c$ and $\delta^{(2)}/\delta j_b \delta j_c$. They are

$$\langle A_2^a(y)\, A_2^b(z)\rangle \;=\; 0 \quad, \tag{4.15}$$

$$\begin{aligned}\langle A_2^a(x)\, \phi^b(y)\rangle \;&=\; \langle \bar{c}^b(y)\, c^a(x)\rangle \\ &=\; -\delta^{ab}\left[\theta(y^1-x^1)-\frac{1}{2}\right]\delta(x^2-y^2)\quad,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\langle b^a(x)\, A_2^b(y)\rangle \;&=\; \partial_2 \langle \bar{c}^a(x)\, c^b(y)\rangle \\ &=\; -\delta^{ab}\left[\theta(x^1-y^1)-\frac{1}{2}\right]\partial_2\delta(x^2-y^2)\quad,\end{aligned} \tag{4.17}$$

where (4.5) and (4.6) have been used.

For the higher orders, (4.15, 4.16, 4.17) generalize to

$$\left\langle A_2^a(y)\, A_2^b(z)\,(\varphi)^n\right\rangle \;=\; 0$$

$$\left\langle A_2^a(x)\, \phi^b(y)\,(\varphi)^n\right\rangle \;=\; \left\langle \bar{c}^b(y)\, c^a(x)\,(\varphi)^n\right\rangle$$

$$\left\langle b^a(x)\, A_2^b(y)\,(\varphi)^n\right\rangle \;=\; \partial_2\left\langle \bar{c}^a(x)\, c^b(y)\,(\varphi)^n\right\rangle$$

with $\varphi \in \{A_2, \phi, c, \bar{c}, b\}$. Since we have already the complete solution for the ghost sector (4.5 ,4.11 4.14), this proves that in the axial gauge the supersymmetry completely fixes all the Green's functions which contain at least one field A_2.

4.3.1 Solution of the Local Gauge Ward–identity

The solution for $\langle (b)^n\rangle$, $\forall n$ can be derived from the local gauge Ward–identity (3.29). Indeed, by differentiation with respect to $\delta^{(n)}/(\delta j_b)^n$, $n \geq 1$, one gets directly

$$\langle (b)^{(n+1)}\rangle = 0 \qquad \forall n \geq 1 \tag{4.18}$$

Since these are the only loop diagrams of the theory, this shows that the non–linearity of (3.33) have nn consequences and that the three approximation (3.6) is exact.

4.3.2 Solution of the field equation for ϕ

In the last two subsections, we showed that all the Green's function of the form

$$\langle A_2 \ldots\rangle \quad, \quad \langle b^n\rangle$$

where fixed by symmetry requirements. The remaining part formed by the Green's functions of the type $\langle (\phi)^m (b)^n\rangle$, $m \geq 1$ is solved only through the use of the

14

equations of motion (3.31). In the following, we will thus look for the general solution of (3.31) for Green's functions with no A_2 fields since the latter are already found in the previous subsection.

For the propagators, (3.31) gives

$$\varepsilon^{\mu\nu}\partial_\nu \left\langle \phi^a(x)b^b(y)\right\rangle = 0 \tag{4.19}$$

$$\partial_1 \langle\phi^a(x)\phi^b(y)\rangle = 0 \tag{4.20}$$

$$\langle b^b(x)\phi^a(y)\rangle = \partial_2 \langle\phi^a(x)\phi^b(y)\rangle \tag{4.21}$$

which, together with (4.18) and translational invariance solve into

$$\langle\phi^a(x)\phi^b(y)\rangle = F(x^2 - y^2) \tag{4.22}$$

$$\langle b^b(x)\phi^a(y)\rangle = 0 \tag{4.23}$$

Here $F(x^2 - y^2)$ is an arbitrary function with canonical dimension 0. Following the same reasoning as for (4.5), the latter is a constant

$$\left\langle \phi^a(x)\phi^b(y)\right\rangle = \text{const } \delta^{ab} \tag{4.24}$$

It is important to notice that this constitute the first solution of the homogeneous equation which is not anihilated by the residual gauge invariance. This is caused by the bosonic character of the field ϕ of canonical dimension 0.

The higher orders are generated by functional differentiating (3.32) with respect to $\delta^{(m+n)}/\delta(j_b)^m \delta(j_\phi)^n$

$$\partial_1 \left\langle \phi^a(x)b(y_1)^{b_1} \ldots b(y_m)^{b_m}\phi(z_1)^{c_1} \ldots \phi(z_n)^{c_n}\right\rangle = \tag{4.25}$$

$$= \sum_{i=1}^m f^{ab_i e}\delta^{(2)}(x - y_i) \left\langle \phi^e(y_i)b(y_1))^{b_1} \ldots b(\widehat{y_i}))^{b_i} \ldots b(y_m)^{b_m}\phi(z_1)^{c_1} \ldots \phi(z_n)^{c_n}\right\rangle$$

For the $m = 0$ case, the solution which generalizes (4.22) is

$$F\left(\ln(\frac{x - z_i}{x - z_j})\right) \tag{4.26}$$

but then (3.33) imposes[7]

$$\langle(\phi)^n\rangle = \beta_n \qquad \forall n \tag{4.27}$$

where β_n is a constant which may depends on n. Physically this correspond to the only invariants polynomials $\text{Tr } \phi^n$.

Furthermore, these solutions are the starting points for the reccurence (4.25) for $m \neq 0$. Nevertheless, the Green's functions obtained by this way do not satisfy the residual gauge invariance (3.30) and we must set $\beta_n = 0$.

---

[7]See footnote [6] on p. 12.

5   Conclusion

We already emphisize that the main difference of the two dimensional BF model with respect to the higher dimensional cases is the absence of reducible symmetry caused by the 0–form nature of the field ϕ. The system is thus less constrained, *i.e.*, the symmetries do not fix all the Green's functions, the monomials $\mathrm{Tr}\,(\phi)^n$ remains free.

References

[1] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, Phys. Reports **209** (1991) 129.

[2] S. Donaldson, *An application of gauge theory to four dimensional topology*, J. Diff. Geom. **18** (1983) 279.

[3] S. Donaldson, *Polynomial invariants of smooth four–manifolds*, Topology **29** (1990) 257.

[4] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988) 353.

[5] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989) 351.

[6] C. Lucchesi, O. Piguet and S.P. Sorella, *Renormalization and finiteness of topological BF theories*, Nucl. Phys. **B395** (1993) 325.

[7] N. Maggiore and S.P. Sorella, *Perturbation theory for antisymmetric tensor fields in four dimensions*, Int. J. Mod. Phys. **A8** (1993) 929.

[8] M. Blau and G. Thompson, *Topological gauge theories of antisymmetric tensor fields*, Ann. Phys. (N.Y.) **205** (1991) 130.

[9] A. Brandhuber, M. Langer, M. Schweda, S.P. Sorella, S. Emery and O. Piguet *Symmetries of the Chern–Simons theory in the axial gauge*, Helv. Phys. Acta **66** (1993) 551.

[10] S. Emery, A. Brandhuber, K. Landsteiner and M. Schweda, *The Three–Dimensional BF Model with Cosmological Term in the Axial Gauge*, Helv. Phys. Acta **68** (1995) 126.

[11] A. Blasi, N. Maggiore, *Infrared and ultraviolet finiteness of topological BF theory in two dimensions*, Class. Quantum Grav. **10**(1993)37.

[12] F. Delduc, F. Gieres and S.P. Sorella, *Supersymmetry of the d = 3 Chern–Simons action in the Landau gauge*, Phys. Lett. **B225** (1989) 367.

[13] C. Becchi, A. Rouet and R. Stora, *Renormalization of gauge theories*, Ann. Phys. (N.Y.) **98** (1976) 287.

[14] I.V. Tyutin, *Gauge invariance in field theory and in statistical physics in the operator formulation*, Lebedev preprint, FIAN No. 39, unpublished.

[15] J. Wess and B. Zumino, Nucl. Phys. **B78** (1974) 1.

[16] J. Wess and B. Zumino, Nucl. Phys. **B70** (1974) 39.

[17] S.P. Sorella, , Comm. Math. Phys. **157** (1993) 231.

[18] A. Blasi, O. Piguet and S.P. Sorella, *Landau gauge and finiteness*, Nucl. Phys. **B356** (1991) 154.

[19] A. Blasi and R. Collina, *Chern–Simons and two–dimensional conformal field theories in a covariant Lagrangian approach*, Phys. Lett. **B243** (1990) 99.

[20] N. Maggiore and S.P. Sorella, *Finiteness of the topological models in the Landau gauge*, Nucl. Phys. **B377** (1992) 236.